ON THE THEORY OF HYPERSONIC GAS FLOW WITH A POWER-LAW SHOCK WAVE

(K TEORII GIPERZVUKOVYKH TECHENII GAZA SO SKACHKAMI UPLOTNENIIA STEPENNOI FORMY)

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V.V. SYCHEV
(Moscow)

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Plane and axisymmetric hypersonic gas flows are considered with shock waves of very great intensity that have a power-law form. On the basis of an investigation of the portions of the flow with high entropy adjoining the surface of the body (not necessarily for a shock wave of the given form) it is shown that the use in the flow problem of the exact solution for the corresponding unsteady self-similar gas motion requires a supplementary refinement of the thickness of the high entropy layer. A method is shown for introducing such a correction and constructing the shape of the body contour, on which is to be applied the pressure distribution obtained on the basis of the theory of small disturbances.

1. According to the theory of small disturbances in a hypersonic stream the problem of flow past a plane or axisymmetric body of small thickness ratio is equivalent to the problem of one-dimensional unsteady gas motion under the action of a plane or cylindrical piston [1]. In this analogy the class of self-similar motions with very intense shock waves propagating according to a power law [2] corresponds to a class of steady flows with shock waves of power-law form

$$y = Cx^n \tag{1.1}$$

with the Mach number of the undisturbed stream $M_\infty \to \infty$. Values of the exponent n lying in the interval $2/(3+\nu) < n < 1$, where $\nu = 0$ for plane and $\nu = 1$ for axisymmetric cases, correspond to flows past convex bodies of power-law form [3]

$$y = cx^n \tag{1.2}$$

The case $n = 2/(3+\nu)$ is singular and corresponds to the problem of a strong explosion [2]. Since $c/C = 0$ in this case, its interpretation as a flow problem consists in the assumption of a finite drag force acting on the leading edge of a body of vanishing thickness. In other words, there is an analogy between the appearance of a strong explosion and the effect of blunting the leading edge of a slender body at large distances from the bluntness [4,5,6].

2. The theory of small disturbances of a hypersonic stream is invalid in the neighborhood of the vertex of the shock wave (1.1), since for $n < 1$ the velocity disturbances are finite there. The value of the entropy on stream lines that intersect the shock wave in that region varies rapidly, and on the surface of the body ($\psi = 0$) the entropy becomes infinitely large, so that the density there becomes equal to zero. As a result, the region of inapplicability of small-disturbance theory comprises all high-entropy portions of the flow, bordering the surface of the body. We consider in more detail the flow in that region.

3. The equations of plane or axisymmetric gas flow, after the von Mises transformation from independent variables x, y to independent variables x, ψ (where ψ is the stream function), can be written in the form

$$\frac{\partial p}{\partial \psi} = -\frac{1}{y^{\nu}}\frac{\partial v}{\partial x} \qquad (3.1)$$

$$\frac{\partial}{\partial x}\frac{p}{\rho^{\gamma}} = 0 \qquad (3.2)$$

$$\frac{\partial y}{\partial \psi} = \frac{1}{\rho u y^{\nu}} \qquad (3.3)$$

$$\frac{\partial y}{\partial x} = \frac{v}{u} \qquad (3.4)$$

$$u^2 + v^2 + \frac{2\gamma}{\gamma - 1}\frac{p}{\rho} = U_{\infty}^{2} \qquad (3.5)$$

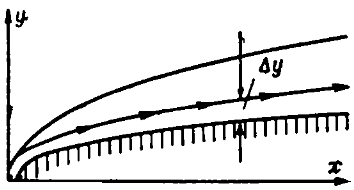

Fig. 1.

Here u, v are the components of the velocity vector, p the pressure, ρ the density, and γ the ratio of specific heats of the gas. Here and henceforth the index ∞ refers to conditions in the undisturbed stream, where we neglect the static pressure. The boundary conditions on the shock-wave surface (1.1) have the form

$$\psi = \frac{1}{1+\nu}\rho_{\infty}U_{\infty}C^{1+\nu}x^{n(1+\nu)},$$

$$p = \frac{2}{\gamma+1}\rho_{\infty}U_{\infty}^{2}\frac{n^2C^2x^{2n-2}}{1+n^2C^2x^{2n-2}}, \qquad \rho = \frac{\gamma+1}{\gamma-1}\rho_{\infty} \qquad (3.6)$$

$$u = U_{\infty}\left(1 - \frac{2}{\gamma+1}\frac{n^2C^2x^{2n-2}}{1+n^2C^2x^{2n-2}}\right), \qquad v = \frac{2}{\gamma+1}U_{\infty}\frac{nCx^{n-1}}{1+n^2C^2x^{2n-2}}$$

The corresponding relations in the small-disturbance theory are obtained by taking $u \approx U_{\infty}$ in Equations (3.1)-(3.4) and taking $1 + n^2C^2x^{2n-1} \approx 1$ in the boundary conditions (3.6) for ψ, v, p and ρ.

4. The exact relations of Paragraph 3 are used for an estimate of the pressure change across the entropy layer at the surface of the body. We define this layer as the flow region comprising the stream lines that intersect the shock wave surface near its vertex, where the angle of inclination of the surface to the free-stream direction is not small

$$y' = \frac{nC}{x^{1-n}} \gtrsim 1 \qquad (4.1)$$

This condition together with (3.6) for ψ gives the following estimate for the change in stream function across the entropy layer

$$\Delta\psi \sim \rho_\infty U_\infty C^{\frac{1+\nu}{1-n}} \qquad (4.2)$$

At sufficiently large distances from the leading edge of the body the transverse velocity component v is proportional to the local angle of inclination of the shock surface (τ), and the pressure to the square of this angle:

$$v \sim U_\infty \tau, \qquad p \sim \rho_\infty U_\infty{}^2 \tau^2 \qquad \left(\tau \sim \frac{C}{x^{1-n}}\right) \qquad (4.3)$$

Substituting these estimates into Equation (3.1) we find that the relative change in pressure across the entropy layer is

$$\frac{\Delta p}{p} \sim \tau^{\frac{n}{1-n}(1+\nu)}$$

Therefore

$$\frac{\Delta p}{p} \lesssim \tau^2 \qquad \text{for } n \geqslant \frac{2}{3+\nu} \qquad (4.4)$$

Since, as is easily seen, the small-disturbance theory involves just the same limit of accuracy, the change in pressure can be neglected.

Hence it follows that the relationship $p(x, \psi)$ obtained on the basis of this theory (but not, as we see, $p(x, y)$) is, for the range of n under consideration, valid in the whole flow field except for the neighborhood of the vertex of the shock wave.

5. For an estimate of the relative thickness of the entropy layer, Equations (3.2) and (3.3) are used. From the condition of constance of entropy along stream lines (3.2) and the boundary conditions (3.6) for p and ρ we find that along the entire entropy layer

$$\frac{p}{\rho^\gamma} \sim \frac{\rho_\infty U_\infty{}^2}{(K\rho_\infty)^\gamma} \qquad \left(K = \frac{\gamma+1}{\gamma-1}\right) \qquad (5.1)$$

Using the estimate (4.3) for the pressure p, we obtain the following estimate for the density

$$\rho \sim K\rho_\infty \tau^{\frac{2}{\gamma}} \tag{5.2}$$

Taking $u \sim U_\infty$ and using (3.3) we find that the relative thickness of the entropy layer is*

$$\frac{\Delta y}{y} \sim \frac{1}{K}\tau^{\frac{n(1+\nu)}{1-n}-\frac{2}{\gamma}} \tag{5.3}$$

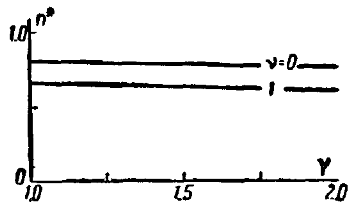

Fig. 2.

It is negligibly small only for

$$n \geqslant n^* = \frac{2+2/\gamma}{3+\nu+2/\gamma} \tag{5.4}$$

This means that in the range of values of γ of practical interest there exists a definite interval of values of the exponent n, $2/(3+\nu) \leqslant n \leqslant n^*$ (Fig. 2), where proper consideration of the thickness of the entropy layer is necessary, for example, in determining the body contour for a given shape of shock wave. Then in the solution of such a problem on the basis of the approximate small-disturbance theory, the region of entropy effect often proves to be thick and the corresponding body contour is found with large error. Thus for determining the relative thickness of the entropy layer we use in this case, together with the condition (3.6) for p, the approximate relation on the surface of the shock wave of the form

$$p = \frac{2}{\gamma+1}\rho_\infty U_\infty{}^2 n^2 C^2 x^{2n-2} \tag{5.5}$$

This, together with (5.2), leads to the following estimate for the density

$$\rho \sim K\rho_\infty \tau^{\frac{2}{\gamma}} C^{-\frac{2}{n\gamma}} \left(\frac{\psi}{\rho_\infty U_\infty}\right)^{\frac{2-2n}{n(1+\nu)\gamma}} \tag{5.6}$$

The result of integrating Equation (3.3) across the entropy layer is

$$\frac{\Delta y}{y} \sim \frac{N}{K}\tau^{\frac{n(1+\nu)}{1-n}-\frac{2}{\gamma}} \tag{5.7}$$

where

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* Estimates analogous to (4.4) and (5.3) for the case $\nu = 0$ were recently obtained in the work [8].

$$N = \left(1 - \frac{2}{\gamma}\frac{1-n}{n(1+\nu)}\right)^{-1} \tag{5.8}$$

The factor N increases with reduction of γ and n (Fig. 3), so that the thickness of the entropy layer determined on the basis of small-disturbance theory may in many cases exceed its actual thickness.

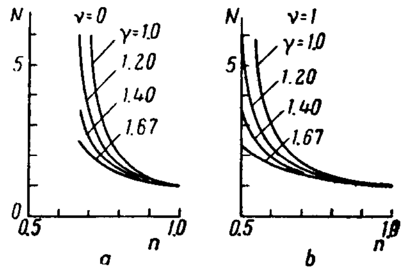

Fig. 3.

6. Thus the use of exact solutions for self-similar gas motion [2] in the case of flows with $n < n^*$ requires special refinement of the entropy layer which (for a given shock wave shape) should lead to a corresponding correction of the contour of the body. The pressure distribution on the surface of the body obtained with the use of exact self-similar solutions should, in accordance with paragraph 4, be applied to the new body contour. The practical determination of the body shape corresponding to a given shock wave shape (3.1) is conveniently carried out by integrating Equation (3.3) across the entire flow field for a series of fixed values of x. In so doing the function $p(x, \psi)$ is known from the self-similar solution for a given shock wave, and the relationship

$$\frac{p}{\rho^{\gamma}} = \varphi(\psi) \tag{6.1}$$

is found from the exact boundary conditions (3.6) for ψ, ρ and p. The velocity vector component u appearing in (3.3), which was above assumed to be $u \sim U_\infty$ in carrying out estimates, must now be made more precise. Since everywhere except in the neighborhood of the vertex of the shock wave $v/u \sim \tau$, we have on the basis of the energy Equation (3.5)

$$u \approx \sqrt{U_\infty^2 - \frac{2\gamma}{\gamma-1}\frac{p}{\rho}} \tag{6.2}$$

with a relative error of order τ^2. Substituting here the estimates obtained previously (4.3) for p and (5.2) for ρ, we obtain

$$\left(\frac{u}{U_\infty}\right)^2 - 1 \sim \tau^{2-\frac{2}{\gamma}} \tag{6.3}$$

Hence follows the incorrectness of the hypersonic equivalence principle ("law of plane sections") in the entropy layer, and the necessity of using Formula (6.2) for the determination of u (with the necessary degree of accuracy). This equation completes the system of relations necessary for the solution.

7. The usual results of integration of the equations of self-similar gas motion appear in the form of the relations

$$v = v_s(x) f(\lambda), \quad \rho = \rho_s g(\lambda), \quad p = p_s(x) h(\lambda) \qquad \left(\lambda = \frac{y}{y_s}\right) \tag{7.1}$$

where index s refers to conditions on the shock wave. The stream function of the self-similar motion necessary for the calculation, which satisfies the differential equation

$$U_\infty \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} = 0 \tag{7.2}$$

is easily found in the form

$$\psi = \psi_s(x) \eta(\lambda) \qquad \eta(\lambda) = \exp\left\{-(1+\nu)\int_1^\lambda \frac{d\lambda}{\frac{2}{\gamma+1} f(\lambda) - \lambda}\right\}$$

This determines the function $p(x, \eta)$ which we can now write, using (5.5), in the form

$$p = \frac{2}{\gamma+1} \rho_\infty U_\infty^2 n^2 C^2 \frac{H(\eta)}{x^{2(1-n)}} \tag{7.4}$$

The entropy distribution function is found from the conditions (3.6) for ψ, ρ and p and may, with consideration of (7.3), but put in the form

$$\frac{p}{\rho^\gamma} = \left(\frac{\gamma-1}{\gamma+1} \frac{1}{\rho_\infty}\right)^\gamma \frac{2}{\gamma+1} \rho_\infty U_\infty^2 n^2 C^2 \left[x^{2(1-n)} \eta^{\frac{2(1-n)}{n(1+\nu)}} + n^2 C^2\right]^{-1} \tag{7.5}$$

Eliminating p from the last two equations we find ρ, and then using Equation (6.2) we determine u; then we obtain

$$\frac{\rho_\infty}{\rho} = \frac{\gamma-1}{\gamma+1} G(x, \eta), \qquad \frac{u}{U_\infty} = \left[1 - \frac{4\gamma}{(\gamma+1)^2} n^2 C^2 \frac{H(\eta) G(x, \eta)}{x^{2(1-n)}}\right]^{\frac{1}{2}} \tag{7.6}$$

where

$$G(x, \eta) = \left[\frac{x^{2(1-n)}}{H(\eta)} \left(x^{2(1-n)} \eta^{\frac{2(1-n)}{n(1+\nu)}} + n^2 C^2\right)^{-1}\right]^{\frac{1}{\gamma}} \tag{7.7}$$

With the replacement of the independent variable ψ by η, Equation (3.3) is put in the form

$$y^{\nu}\frac{\partial y}{\partial \eta}=\frac{\psi_s(x)}{\rho u} \tag{7.8}$$

Substituting here the determined functions $\rho(x, \eta)$, $u(x, \eta)$ and integrating for fixed x, we obtain an equation determining the shape and location of the stream lines in the flow under consideration:

$$y(x, \eta)=Gx^n\left\{1+\frac{\gamma-1}{\gamma+1}\int_1^{\eta}G(x, \eta)\left(1-\frac{4\gamma}{(\gamma+1)^2}n^2C^2\frac{H(\eta)\,G(x, \eta)}{x^{2(1-n)}}\right)^{-\frac{1}{2}}d\eta\right\}^{\frac{1}{1+\nu}} \tag{7.9}$$

The new contour of the body is determined by this equation if the upper limit of integration is set equal to zero.

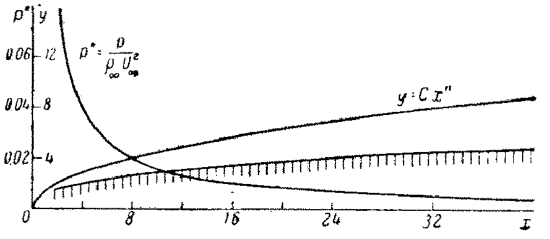

Fig. 4.

8. As an example of application of the result obtained we consider axisymmetric flow with a shock wave of the form

$$y=\sqrt{2x} \tag{8.1}$$

The calculations were carried out for the case $\gamma = 1.4$. The functions (7.1) were taken from the tables given in the work [2]. The results of the calculations are given in Figs. 4 and 5. From consideration of the

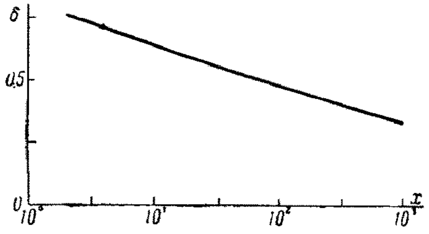

Fig. 5.

relations obtained it is evident that the relative thickness of the body contour determined on the basis of Equation (7.9)

$$\delta(x)=\frac{y(x, 0)}{y(x, 1)} \tag{8.2}$$

is negligible only in the region of very large values of x (as a unit of measure for x we take the radius of curvature of the shock wave at its vertex).

Thus the pressure distribution ascribed on the basis of the analogy with a strong explosion to a cylinder with a blunted nose section is in actuality realized on a body of significantly large relative thickness. It appears that this may explain the serious quantitative disagreement thus obtained for the pressure distribution on the surface of a cylinder

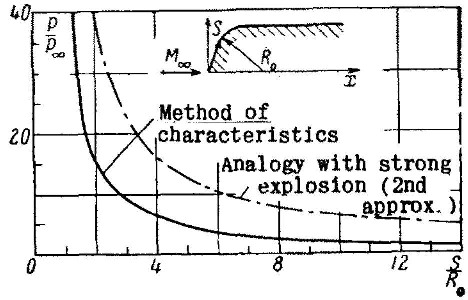

Fig. 6.

with a hemispherical nose section with the results of numerical calculations [7] for a Mach number $M_\infty = 20$, reproduced in Fig. 6, where the abscissa is the relative arc length along the body, measured from the critical point.

In conclusion we note that the self-similar solutions in all cases certainly retain their significance as asymptotic representations of the exact solution for $x \to \infty$.

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